Differential Equations in Random Domains

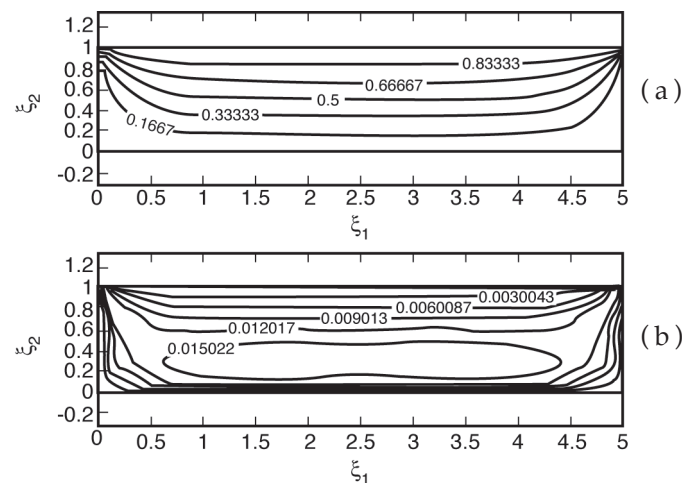*Daniel M. Tartakovsky, T-7; and Dongbin Xiu, Purdue University*

Physical phenomena in domains with rough boundaries play an important role in a variety of applications ranging from surface imaging to manufacturing of nanodevices. Often the topology of such boundaries cannot be accurately described in all of its relevant details due to either insufficient data, or measurement errors, or both. In such cases, this topological uncertainty can be efficiently handled by treating rough boundaries as random fields, so that an underlying physical phenomenon is described by deterministic or stochastic differential equations in random domains. To deal with this class of problems, we developed a novel computational framework, which is based on stochastic mappings to transform the original deterministic/stochastic problem in a random domain into a stochastic problem in a deterministic domain. The latter problem has been studied more extensively, and existing analytical/numerical techniques can be readily applied.

Given a proper spatial resolution, virtually any natural or manufactured surface becomes rough. Consequently, there is a growing interest in experimental, theoretical, and numerical studies of deterministic and probabilistic descriptions of such surfaces and of solutions of differential equations defined on the resulting domains.

The early attempts to represent surface roughness and to study its effects on the system behavior were based on simplified, easily parameterizable, deterministic surface inhomogeneities, such as symmetrical asperities to represent indentations and semispheres to represent protrusions. Alternatively, one can use random fields to represent rough surfaces whose detailed topology cannot be ascertained due to the lack of sufficient information and/or measurement errors. We adopted random representations of rough surfaces because of their generality. Such an approach allows one, not only to make predictions of the system behavior, but also to quantify the corresponding predictive uncertainties.

The presence of uncertainty in rough boundaries necessitates the development of new approaches for the analysis and numerical solution of differential equations defined on random domains. For example, it has been demonstrated that classical variational formulations might not be suitable for such problems, and finite difference approaches remain accurate only for relatively simple rectangular irregularities.

Fig. 1.
(a) Predicted concentration distribution, and (b) the corresponding predictive error bounds.

The adoption of a probabilistic framework to describe rough surfaces makes even an essentially deterministic problem stochastic, e.g., deterministic equations in random domains give rise to stochastic boundary-value problems. This necessitates the search for new stochastic analyses and algorithms. For example, one of the very few existing numerical studies has employed traditional Monte Carlo simulations, which have turned out to be so computationally expensive, as to become impractical.

Recently, we presented a computational framework that is applicable to a wide class of deterministic and stochastic differential equations defined on domains with random (rough) boundaries. A key component of this framework is the use of robust stochastic mappings to transform an original deterministic or stochastic differential equation defined on a random domain into a stochastic differential equation defined on a deterministic domain. This allowed us to employ the well-developed theoretical and numerical techniques for solving stochastic differential equations in deterministic domains.

In [1], we analyzed diffusion in the rectangle with the rough random bottom. The geometric uncertainty translates into the predictive uncertainty. Hence, the best estimate of the concentration of a diffusing substance must be accompanied by a measure of the corresponding predictive uncertainty. This is accomplished by computing the mean and standard deviation of concentration, both of which are shown in Fig. 1.

In [2], we conducted a similar analysis of transport in Stockes flow in a tube with a rough surface (see Fig. 2). We found that for low to moderate roughness (the normalized standard deviation of the surface roughness below 5%), its effects on dispersion of a passive scalar are negligible, so that one can employ standard deterministic models that are much less computationally intensive.

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[1] D. Xiu and D.M. Tartakovsky, "Numerical Methods for Differential Equations in Random Domains," SIAM *J. Sci. Comput.,* (under review) 2005.
[2] D.M. Tartakovsky and D. Xiu, "Stochastic Analysis of Transport in Tubes with Rough Walls," *J. Comp. Phys.,* (under review) 2005.
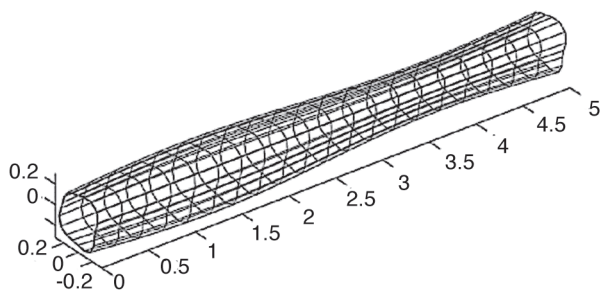
Fig. 2.
A schematic representation of a tube with a rough surface.